Kuwait University — Math 102 — Date: April 11, 2010
Dept of Math and Comp Sci — First Midterm — Duration: 90 minutes

Answer all of the questions. **Calculators, pagers and mobile telephones are NOT allowed.**

1. (4 pts) Let $f(x) = \dfrac{e^x}{1+2e^x}$, $x \leq 0$.

a) Show that f is one-to-one.

b) Find $f^{-1}(x)$ and the domain of f^{-1}.

2. (3 pts) Find all x for which

$$\log_2(x+1) = 1 + \log_{1/2}(x+2).$$

3. (3 pts) Show that $\sin(2\tan^{-1} x) = \dfrac{2x}{1+x^2}$ for all $x \in \mathbb{R}$.

4. (3 pts) Find $\frac{dy}{dx}$ if

$$y = \left(3^{-x^2}\right)\left(\frac{\tanh x}{\sin^{-1} x}\right)^{x/2}, \quad 0 < x < 1.$$

5. (3+3 pts) Evaluate each integral.

a) $\displaystyle\int \frac{\sec^2 x}{4+3\tan x}\,dx.$

b) $\displaystyle\int \frac{\cosh x}{4\cosh^2 x + 5}\,dx.$

6. (3+3 pts) Evaluate the limit, if it exists.

a) $\displaystyle\lim_{x\to 0} \frac{e^{-x^2}-1}{x \sin x}.$

b) $\displaystyle\lim_{x\to 1^+} \left(\frac{4}{\pi}\tan^{-1} x\right)^{1/(\ln x)}.$

SOLUTIONS

1. a. (1.5p). $f'(x) = \frac{e^x}{(1+2e^x)^2} > 0 \Rightarrow f$ is increasing on $(-\infty, 0] \Rightarrow f$ is one-to-one on $(-\infty, 0]$.

b. (2.5p). Put $y = \frac{e^x}{1+2e^x}$ and deduce that $e^x = \frac{y}{1-2y}$. Apply „ln" to both sides and obtain $x = \ln\left(\frac{y}{1-2y}\right)$. So, the inverse function is $f^{-1}(x) = \ln\left(\frac{x}{1-2x}\right)$. Calculate: $\lim\limits_{x\to-\infty} \frac{e^x}{1+2e^x} = 0$ and $\lim\limits_{x\to 0} \frac{e^x}{1+2e^x} = \frac{1}{3}$. So, $D_{f^{-1}} = R_f = (0, \frac{1}{3}]$.

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2. Put the conditions: $x+1>0$, $x+2>0$ and obtain $x \in (-1, \infty)$. Then use the formula $\log_a x = \frac{\ln x}{\ln a}$ and obtain $\ln(x+1)(x+2) = \ln 2 \Leftrightarrow (x+1)(x+2) = 2 \Leftrightarrow$ $x^2+3x = 0$ on $(-1, \infty)$. Thus $x = 0$ is the only solution of the equation.

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3. Put $\tan^{-1}x = y$ and deduce that $x = \tan y$. Then calculate both sides as functions of y: $\sin(2\tan^{-1}x) = \sin 2y$, and $\frac{2x}{1+x^2} = \frac{2\tan y}{1+\tan^2 y} = \frac{2\tan y}{\sec^2 y} = 2\sin y \cos y$. As $\sin 2y = 2\sin y\cos y$, we proved the equality.

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4. Apply „ln" and obtain $\ln y = -x^2\ln 3 + \frac{x}{2}\left(\ln(\tanh x) - \ln(\sin^{-1}x)\right)$. Then take derivatives about x in both sides and deduce that

$$y' = y\left\{-2x\ln 3 + \frac{1}{2}\ln\left(\frac{\tanh x}{\sin^{-1}x}\right) + \frac{x}{2}\left(\operatorname{sech} x \operatorname{csch} x - \frac{1}{\sqrt{1-x^2}\,\sin^{-1}x}\right)\right\}.$$

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5. a. Substitution: $u = 4+3\tan x \Rightarrow du = 3\sec^2 x\,dx \Rightarrow \sec^2 x\,dx = \frac{1}{3}du$. So:

$$I = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|4+3\tan x| + C.$$

b. Substitution: $u = \sinh x \Rightarrow du = \cosh x\,dx$, so the integral becomes

$$I = \int \frac{\cosh x}{4(1+\sinh^2 x)+5}dx = \int \frac{\cosh x}{4\sinh^2 x + 9}dx = \frac{1}{4}\int \frac{\cosh x}{\sinh^2 x + \frac{9}{4}}dx = \frac{1}{4}\int \frac{du}{u^2 + \left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2u}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{2\sinh x}{3}\right) + C.$$

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6. a. $\lim\limits_{x\to 0} \frac{e^{-x^2}-1}{x\sin x} = \frac{0}{0}$ (Ind. F.). Calculate: $\lim\limits_{x\to 0} \frac{(e^{-x^2}-1)'}{(x\sin x)'} = \lim\limits_{x\to 0} \frac{-2xe^{-x^2}}{\sin x + x\cos x} = \frac{0}{0}$, which is also Ind. F. Calculate: $\lim\limits_{x\to 0} \frac{(-2xe^{-x^2})'}{(\sin x + x\cos x)'} = \lim\limits_{x\to 0} \frac{-2e^{-x^2} + 4x^2e^{-x^2}}{\cos x + \cos x - x\sin x} = -1$.

Thus, by L'Hôpital's Rule we obtain $\lim\limits_{x\to 0} \frac{e^{-x^2}-1}{x\sin x} = -1$.

b. $\lim\limits_{x\to 1^+} \left(\frac{4}{\pi}\tan^{-1}x\right)^{\frac{1}{\ln x}} = 1^\infty$ (Ind. F.). Put $y = \left(\frac{4}{\pi}\tan^{-1}x\right)^{\frac{1}{\ln x}}$ and apply „ln" to both sides: $\ln y = \frac{\ln\left(\frac{4}{\pi}\tan^{-1}x\right)}{\ln x}$. Then $\lim\limits_{x\to 1^+} \ln y = \frac{0}{0}$ (Ind. F).

Calculate: $\lim\limits_{x\to 1^+} \frac{\left(\ln\left(\frac{4}{\pi}\tan^{-1}x\right)\right)'}{(\ln x)'} = \lim\limits_{x\to 1^+} \frac{\frac{1}{(1+x^2)\tan^{-1}x}}{\frac{1}{x}} = \lim\limits_{x\to 1^+} \frac{x}{(1+x^2)\tan^{-1}x} = \frac{2}{\pi}$.

Thus $\lim\limits_{x\to 1^+} \ln y = \frac{2}{\pi}$ and therefore $\lim\limits_{x\to 1^+} y = e^{\frac{2}{\pi}}$.
